

Chapter 8

DEFLECTION OF BEAMS BY INTEGRATION

8.1. INTRODUCTION

We saw in Sec. 4.4 that a prismatic beam subjected to pure bending is bent into an arc of circle and that, within the elastic range, the curvature of the neutral surface may be expressed as

$$\frac{1}{\rho} = \frac{M}{EI} \quad (4.21)$$

where M is the bending moment, E the modulus of elasticity, and I the moment of inertia of the cross section about its neutral axis.

When a beam is subjected to a transverse loading, Eq. (4.21) remains valid for any given transverse section, provided that Saint-Venant's principle applies. However, both the bending moment and the curvature of the neutral surface will vary from section to section. Denoting by x the distance of the section from the left end of the beam, we write

$$\frac{1}{\rho} = \frac{M(x)}{EI} \quad (8.1)$$

Consider, for example, a cantilever beam AB of length L subjected to a concentrated load P at its free end A (Fig. 8.1a). We have $M(x) = -Px$ and, substituting into (8.1),

$$\frac{1}{\rho} = -\frac{Px}{EI}$$

which shows that the curvature of the neutral surface varies linearly with x , from zero at A , where ρ_A itself is infinite, to $-PL/EI$ at B , where $|\rho_B| = EI/PL$ (Fig. 8.1b).

Consider now the overhanging beam AD of Fig. 8.2, which supports two concentrated loads as shown. From the free-body diagram of the beam (Fig. 8.3a), we find that the reactions at the supports are $R_A = 1$ kN and $R_C = 5$ kN, respectively, and draw the corresponding bending-moment

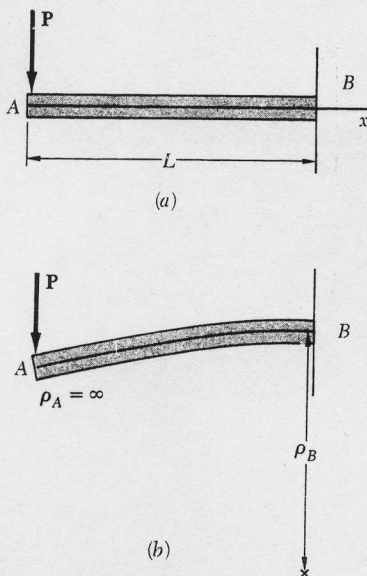


Fig. 8.1

diagram (Fig. 8.3b). We note from the diagram that M , and thus the curvature of the beam, are zero at both ends of the beam, and also at a point E located at $x = 4$ m. Between A and E the bending moment is positive and the beam is concave upward; between E and D the bending moment is negative and the beam is concave downward (Fig. 8.3c). We also note that the largest value of the curvature (i.e., the smallest value of the radius of curvature) occurs at the support C , where $|M|$ is maximum.

From the information obtained on its curvature, we may get a fairly good idea of the shape of the deformed beam. However, the analysis and design of a beam usually require more precise information on the *deflection* and the *slope* of the beam at various points. Of particular importance is the knowledge of the *maximum deflection* of the beam. In this chapter we shall use Eq. (8.1) to obtain a relation between the deflection y measured at a given point Q on the axis of the beam and the distance x of that point from some fixed origin (Fig. 8.4). The relation obtained is the equation of the *elastic curve*, i.e., the equation of the curve into which the axis of the beam is transformed under the given loading (Fig. 8.4b).†

8.2. EQUATION OF THE ELASTIC CURVE

We first recall from elementary calculus that the curvature of a plane curve at a point $Q(x, y)$ of the curve may be expressed as

$$\frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}} \quad (8.2)$$

where dy/dx and d^2y/dx^2 are the first and second derivatives of the function $y(x)$ represented by that curve. But, in the case of the elastic curve of a beam, the slope dy/dx is very small, and its square is negligible compared to unity. We may write, therefore,

$$\frac{1}{\rho} = \frac{d^2y}{dx^2} \quad (8.3)$$

Substituting for $1/\rho$ from (8.3) into (8.1), we have

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI} \quad (8.4)$$

The equation obtained is a second-order linear differential equation; it is the governing differential equation for the elastic curve.

† It should be noted that, in this chapter and the next, y represents a vertical displacement, while it was used in previous chapters to represent the distance of a given point in a transverse section from the neutral axis of that section.

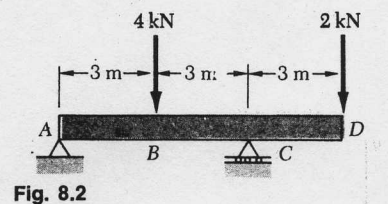


Fig. 8.2

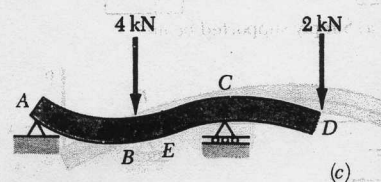
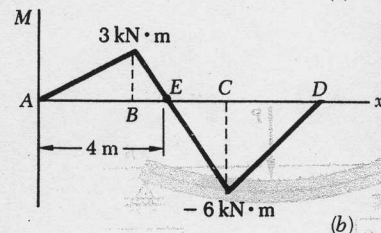
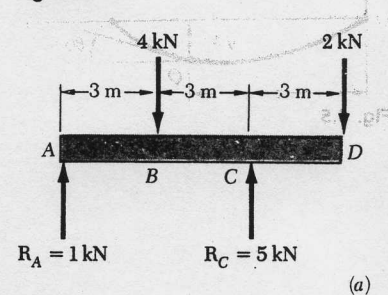


Fig. 8.3

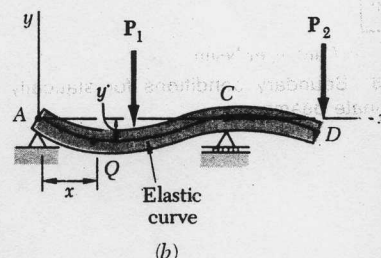
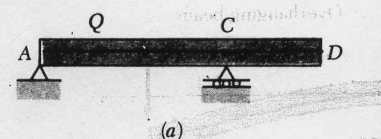


Fig. 8.4

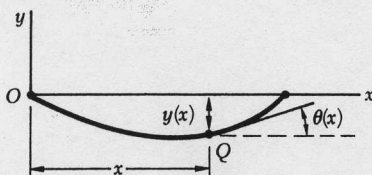


Fig. 8.5

The product EI is known as the *flexural rigidity* and, if it varies along the beam, as in the case of a beam of varying depth, we must express it as a function of x before proceeding to integrate Eq. (8.4). However, in the case of a prismatic beam, which is the case considered here, the flexural rigidity is constant. We may thus multiply both members of Eq. (8.4) by EI and integrate in x . We write

$$EI \frac{dy}{dx} = \int_0^x M(x) dx + C_1 \quad (8.5)$$

where C_1 is a constant of integration. Denoting by $\theta(x)$ the angle, measured in radians, that the tangent at Q to the elastic curve forms with the horizontal (Fig. 8.5), and recalling that this angle is very small, we have

$$\frac{dy}{dx} = \tan \theta \simeq \theta(x)$$

Thus, we may write Eq. (8.5) in the alternate form

$$EI \theta(x) = \int_0^x M(x) dx + C_1 \quad (8.5')$$

Integrating both members of Eq. (8.5) in x , we have

$$EI y = \int_0^x \left[\int_0^x M(x) dx + C_1 \right] dx + C_2$$

$$EI y = \int_0^x dx \int_0^x M(x) dx + C_1 x + C_2 \quad (8.6)$$

where C_2 is a second constant, and where the first term in the right-hand member represents the function of x obtained by integrating twice in x the bending moment $M(x)$. If it were not for the fact that the constants C_1 and C_2 are as yet undetermined, Eq. (8.6) would define the deflection of the beam at any given point Q , and Eq. (8.5) or (8.5') would similarly define the slope of the beam at Q .

The constants C_1 and C_2 are determined from the *boundary conditions* or, more precisely, from the conditions imposed on the beam by its supports. Limiting our analysis in this section to *statically determinate beams*, i.e., to beams supported in such a way that the reactions at the supports may be obtained by the methods of statics, we note that only three types of beams need to be considered here (Fig. 8.6): (a) the *simply supported beam*, (b) the *overhanging beam*, and (c) the *cantilever beam*.

In the first two cases, the supports consist of a pin and bracket at A and of a roller at B , and require that the deflection be zero at each of these points. Letting first $x = x_A$, $y = y_A = 0$ in Eq. (8.6), and then $x = x_B$, $y = y_B = 0$ in the same equation, we obtain two equations which may be solved for C_1 and C_2 . In the case of the cantilever beam (Fig. 8.6c), we note that both the deflection and the slope at A must be zero. Letting $x = x_A$, $y = y_A = 0$ in Eq. (8.6), and $x = x_A$, $\theta = \theta_A = 0$ in Eq. (8.5'), we obtain again two equations which may be solved for C_1 and C_2 .

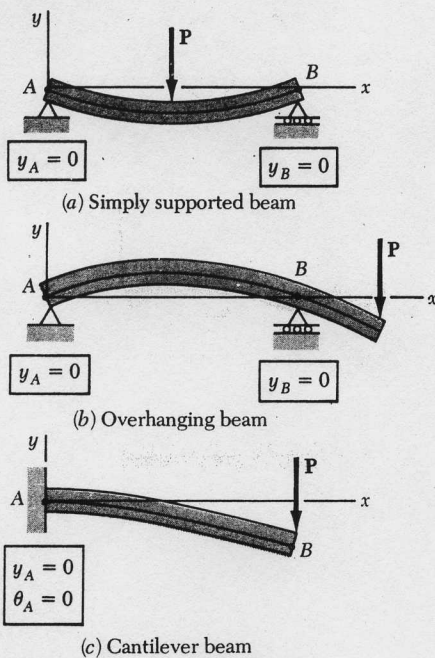


Fig. 8.6 Boundary conditions for statically determinate beams.

Example 8.01

The cantilever beam AB is of uniform cross section and carries a load P at its free end A (Fig. 8.7). Determine the equation of the elastic curve and the deflection and slope at A .

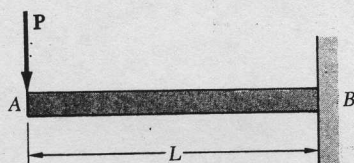


Fig. 8.7

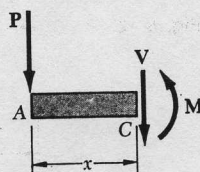


Fig. 8.8

Using the free-body diagram of the portion AC of the beam (Fig. 8.8), where C is located at a distance x from end A , we find

$$M = -Px \quad (8.7)$$

Substituting for M into Eq. (8.4) and multiplying both members by the constant EI , we write

$$EI \frac{d^2y}{dx^2} = -Px$$

Integrating in x , we obtain

$$EI \frac{dy}{dx} = -\frac{1}{2}Px^2 + C_1 \quad (8.8)$$

We now observe that at the fixed end B we have $x = L$ and $\theta = dy/dx = 0$ (Fig. 8.9). Substituting these values into (8.8) and solving for C_1 we have

$$C_1 = \frac{1}{2}PL^2$$

which we carry back into (8.8):

Example 8.02

The simply supported prismatic beam AB carries a uniformly distributed load w per unit length (Fig. 8.10). Determine the equation of the elastic curve and the maximum deflection of the beam.

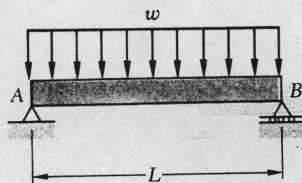
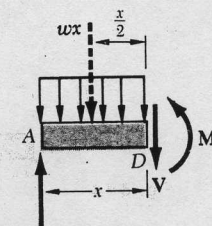


Fig. 8.10



$R_A = \frac{1}{2}wL$
Fig. 8.11

$$EI \frac{dy}{dx} = -\frac{1}{2}Px^2 + \frac{1}{2}PL^2 \quad (8.9)$$

Integrating both members of Eq. (8.9), we write

$$EI y = -\frac{1}{6}Px^3 + \frac{1}{2}PL^2x + C_2 \quad (8.10)$$

But, at B we have $x = L$, $y = 0$. Substituting into (8.10), we have

$$0 = -\frac{1}{6}PL^3 + \frac{1}{2}PL^3 + C_2$$

$$C_2 = -\frac{1}{3}PL^3$$

Carrying the value of C_2 back into Eq. (8.10), we obtain the equation of the elastic curve:

$$EI y = -\frac{1}{6}Px^3 + \frac{1}{2}PL^2x - \frac{1}{3}PL^3$$

or

$$y = \frac{P}{6EI}(-x^3 + 3L^2x - 2L^3) \quad (8.11)$$

The deflection and slope at A are obtained by letting $x = 0$ in Eqs. (8.11) and (8.9). We find

$$y_A = -\frac{PL^3}{3EI} \quad \text{and} \quad \theta_A = \left(\frac{dy}{dx}\right)_A = \frac{PL^2}{2EI}$$

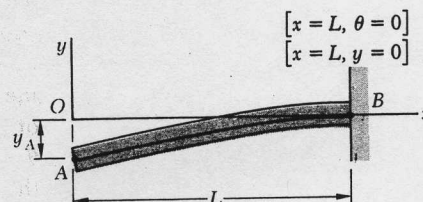


Fig. 8.9

Drawing the free-body diagram of the portion AD of the beam (Fig. 8.11), and taking moments about D , we find that

$$M = \frac{1}{2}wLx - \frac{1}{2}wx^2 \quad (8.12)$$

Substituting for M into Eq. (8.4) and multiplying both members of this equation by the constant EI , we write

$$EI \frac{d^2y}{dx^2} = -\frac{1}{2}wx^2 + \frac{1}{2}wLx \quad (8.13)$$

Integrating twice in x , we have

$$EI \frac{dy}{dx} = -\frac{1}{6}wx^3 + \frac{1}{4}wLx^2 + C_1 \quad (8.14)$$

$$EI y = -\frac{1}{24}wx^4 + \frac{1}{12}wLx^3 + C_1x + C_2 \quad (8.15)$$

Observing that $y = 0$ at both ends of the beam (Fig. 8.12), we first let $x = 0$ and $y = 0$ in Eq. (8.15) and obtain $C_2 = 0$. We then make $x = L$ and $y = 0$ in the same equation and write

$$0 = -\frac{1}{24}wL^4 + \frac{1}{12}wL^4 + C_1L$$

$$C_1 = -\frac{1}{24}wL^3$$

Carrying the values of C_1 and C_2 back into Eq. (8.15), we obtain the equation of the elastic curve:

$$EI y = -\frac{1}{24}wx^4 + \frac{1}{12}wLx^3 - \frac{1}{24}wL^3x$$

or

$$y = \frac{w}{24EI}(-x^4 + 2Lx^3 - L^3x) \quad (8.16)$$

Substituting into Eq. (8.14) the value obtained for C_1 , we check that the slope of the beam is zero for $x = L/2$ and that the elastic curve has a minimum at the midpoint C of the beam (Fig. 8.13). Letting $x = L/2$ in Eq. (8.16), we have

$$y_c = \frac{w}{24EI} \left(-\frac{L^4}{16} + 2L \frac{L^3}{8} - L^3 \frac{L}{2} \right) = -\frac{5wL^4}{384EI}$$

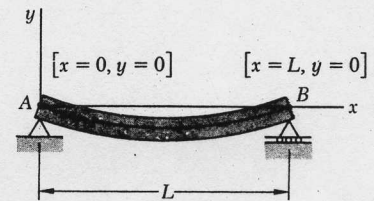


Fig. 8.12

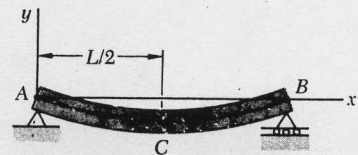


Fig. 8.13

The maximum deflection or, more precisely, the maximum absolute value of the deflection, is thus

$$|y|_{\max} = \frac{5wL^4}{384EI}$$

In each of the two examples considered so far, only one free-body diagram was required to determine the bending moment in the beam. As a result, a single function of x was used to represent M throughout the beam. This, however, is not generally the case. Concentrated loads, reactions at supports, or discontinuities in a distributed load will make it necessary to divide the beam into several portions, and to represent the bending moment by a different function $M(x)$ in each of these portions of beam. Each of the functions $M(x)$ will then lead to a different expression for the slope $\theta(x)$ and for the deflection $y(x)$. Since each of the expressions obtained for the deflection must contain two constants of integration, a large number of constants will have to be determined. As we shall see in the next example, the required additional boundary conditions may be obtained by observing that, while the shear and bending moment can be discontinuous at several points in a beam, the *deflection* and the *slope* of the beam *cannot be discontinuous* at any point.

Example 8.03

For the prismatic beam and the loading shown (Fig. 8.14), determine the slope and deflection at point D .

We must divide the beam into two portions, AD and DB , and determine the function $y(x)$ which defines the elastic curve for each of these portions.

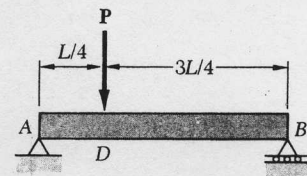


Fig. 8.14

1. **From A to D** ($x < L/4$). We draw the free-body diagram of a portion of beam AE of length $x < L/4$ (Fig. 8.15). Taking moments about E, we have

$$M_1 = \frac{3P}{4}x \quad (8.17)$$

or, recalling Eq. (8.4),

$$EI \frac{d^2y_1}{dx^2} = \frac{3}{4}Px \quad (8.18)$$

where $y_1(x)$ is the function which defines the elastic curve for portion AD of the beam. Integrating in x , we write

$$EI \theta_1 = EI \frac{dy_1}{dx} = \frac{3}{8}Px^2 + C_1 \quad (8.19)$$

$$EI y_1 = \frac{1}{8}Px^3 + C_1x + C_2 \quad (8.20)$$

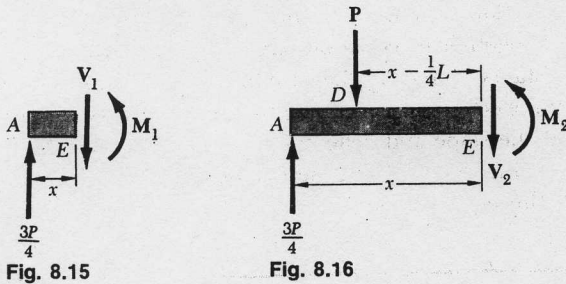


Fig. 8.15

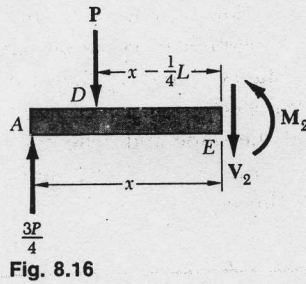


Fig. 8.16

2. **From D to B** ($x > L/4$). We now draw the free-body diagram of a portion of beam DE of length $x > L/4$ (Fig. 8.16) and write

$$M_2 = \frac{3P}{4}x - P\left(x - \frac{L}{4}\right) \quad (8.21)$$

or, recalling Eq. (8.4) and rearranging terms,

$$EI \frac{d^2y_2}{dx^2} = -\frac{1}{4}Px + \frac{1}{4}PL \quad (8.22)$$

where $y_2(x)$ is the function which defines the elastic curve for portion DB of the beam. Integrating in x , we write

$$EI \theta_2 = EI \frac{dy_2}{dx} = -\frac{1}{8}Px^2 + \frac{1}{4}PLx + C_3 \quad (8.23)$$

$$EI y_2 = -\frac{1}{24}Px^3 + \frac{1}{8}PLx^2 + C_3x + C_4 \quad (8.24)$$

Determination of the Constants of Integration. The conditions which must be satisfied by the constants of integration have been summarized in Fig. 8.17. At the support A, where

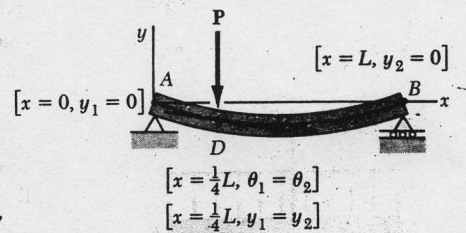


Fig. 8.17

the deflection is defined by Eq. (8.20), we must have $x = 0$ and $y_1 = 0$. At the support B, where the deflection is defined by Eq. (8.24), we must have $x = L$ and $y_2 = 0$. Also, the fact that there can be no sudden change in deflection or in slope at point D requires that $y_1 = y_2$ and $\theta_1 = \theta_2$ when $x = L/4$. We have therefore:

$$[x = 0, y_1 = 0], \text{ Eq. (8.20): } 0 = C_2 \quad (8.25)$$

$$[x = L, y_2 = 0], \text{ Eq. (8.24): } 0 = \frac{1}{12}PL^3 + C_3L + C_4 \quad (8.26)$$

$$[x = L/4, \theta_1 = \theta_2], \text{ Eqs. (8.19) and (8.23):}$$

$$\frac{3}{128}PL^2 + C_1 = \frac{7}{128}PL^2 + C_3 \quad (8.27)$$

$$[x = L/4, y_1 = y_2], \text{ Eqs. (8.20) and (8.24):}$$

$$\frac{PL^3}{512} + C_1 \frac{L}{4} = \frac{11PL^3}{1536} + C_3 \frac{L}{4} + C_4 \quad (8.28)$$

Solving these equations simultaneously, we find

$$C_1 = -\frac{7PL^2}{128}, C_2 = 0, C_3 = -\frac{11PL^2}{128}, C_4 = \frac{PL^3}{384}$$

Substituting for C_1 and C_2 into Eqs. (8.19) and (8.20), we write that, for $x \leq L/4$,

$$EI \theta_1 = \frac{3}{8}Px^2 - \frac{7PL^2}{128} \quad (8.29)$$

$$EI y_1 = \frac{1}{8}Px^3 - \frac{7PL^2}{128}x \quad (8.30)$$

Letting $x = L/4$ in each of these equations, we find that the slope and deflection at point D are, respectively,

$$\theta_D = -\frac{PL^2}{32EI} \quad \text{and} \quad y_D = -\frac{3PL^3}{256EI}$$

We note that, since $\theta_D \neq 0$, the deflection at D is *not* the maximum deflection of the beam.

*8.4. USE OF SINGULARITY FUNCTIONS

Reviewing the work done in the first three sections of this chapter, we note that the integration method provides a convenient and effective way for determining the slope and deflection at any point of a prismatic beam, as long as the bending moment may be represented by a single analytical function $M(x)$. However, when the loading of the beam is such that two different functions are needed to represent the bending moment over the entire length of the beam, as in Example 8.03, four constants of integration are required, and an equal number of equations, expressing continuity conditions at point D , as well as boundary conditions at the supports A and B , must be used to determine these constants. If three or more functions were needed to represent the bending moment, additional constants and a corresponding number of additional equations would be required, resulting in rather lengthy computations. We shall see in this section how the use of *singularity functions* may simplify the computations.

Considering again the beam and loading of Example 8.03 (Fig. 8.14), we recall from Eqs. (8.17) and (8.21) that the bending moment over the portions AD and DB of the beam may be expressed by the functions

$$M_1(x) = \frac{3P}{4}x \quad \left(0 \leq x \leq \frac{L}{4}\right) \quad (8.38)$$

$$M_2(x) = \frac{3P}{4}x - P\left(x - \frac{L}{4}\right) \quad \left(\frac{L}{4} \leq x \leq L\right) \quad (8.39)$$

where x is the distance measured from end A . The functions $M_1(x)$ and $M_2(x)$ may be represented by the single expression

$$M(x) = \frac{3P}{4}x - P\left\langle x - \frac{L}{4} \right\rangle \quad (8.40)$$

if we specify that the second term should be included in our computations when $x \geq L/4$, and ignored when $x < L/4$. In other words, the brackets $\langle \rangle$ should be replaced by ordinary parentheses $()$ when $x \geq L/4$, and by zero when $x < L/4$.

Substituting for $M(x)$ from (8.40) into Eq. (8.4), we write

$$EI \frac{d^2y}{dx^2} = \frac{3P}{4}x - P\left\langle x - \frac{L}{4} \right\rangle \quad (8.41)$$

and, integrating in x ,

$$EI \theta = EI \frac{dy}{dx} = \frac{3}{8}Px^2 - \frac{1}{2}P\left\langle x - \frac{L}{4} \right\rangle^2 + C_1 \quad (8.42)$$

$$EI y = \frac{1}{8}Px^3 - \frac{1}{6}P\left\langle x - \frac{L}{4} \right\rangle^3 + C_1x + C_2 \quad (8.43)$$

where, again, the brackets should be replaced by parentheses when $x \geq L/4$, and by zero when $x < L/4$.

The constants C_1 and C_2 may be determined from the boundary condi-

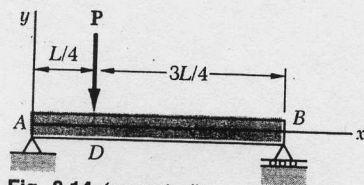


Fig. 8.14 (repeated)

