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Refereed paper

Electro, thermal and elastic characterizations of suspended micro beams

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An electro-thermal-elastic model for characterizing thermal buckling behavior of micromachined beams has been developed. These micro beams are made of heavily phosphorus-doped polysilicon by surface micromachining technology. When electrically self-heated, the clamped–clamped beams may buckle under high compressive thermal stresses. The electro-thermal and thermal-elastic responses have been analyzed separately to establish the overall electro-thermal-elastic characteristics. It is predicted that a 2 μm wide, 2.2 μm thick and 100 μm long clamped–clamped suspended micro beam will buckle under a 6 mA input current. The lateral deflection is about 2.7 μm at the center of the beam. The analytical results are consistent with experimental observations. © 1998 Published by Elsevier Science Ltd. All rights reserved.

Nomenclature

A	Cross-section area of micro beams
B_{ij}	Numerical constants
E	Young's modulus of polysilicon
$E(\beta)$	The complete elliptic integral of the second kind
I	Moment of inertial of micro beams
$K(\beta)$	Complete elliptic integral of the first kind
L	Length of the beam before deformation
L'	Length of the beam after deformation
M_a	Reaction moment at clamped edges of a beam
P	Thermal load
s	Coordinate along the beam
T	Temperature along the micro beam
T_0	A reference temperature
T_{avg}	Average temperature of a micro beam
T_r	A variable term in heat equation
T_∞	Ambient (bulk) temperature
x	Coordinate along the axial direction
y	Coordinate along the lateral direction
y_{max}	Maximum deflection of the beam
α	Thermal expansion coefficient
β	Variable in substitution for θ_{max}
δ_{ij}	Kronecker delta
ε	A combined variable in heat equation
ε_{ij}	Strain components
θ	Deflection angle of the beam

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θ_{max}	Maximum deflection angle of the beam
λ	Lamè constant
μ	Lamè constant
v	Poisson's ratio
τ_{ij}	Stress components
ϕ	$\sin^{-1}\left(\frac{\sin(\theta/2)}{\beta}\right)$

1. Introduction

Buckling phenomena of micro beams have found several applications in microsystems. For example, buckling criteria of micro beams have been used to predict the magnitude of residual stress in thin films [1,2]. Different bistable or multistable mechanisms of buckled beams have been applied for memory elements [3], logic elements [4], pumping mechanism [5], and a snapping microactuator [6]. It is noted that all of the above applications have used the on/off mechanisms of the buckling effects.

This paper presents electro, thermal and elastic modeling of the continuous buckling mechanism for suspended micromachined beams. These micro beams are made of heavily phosphorus-doped polysilicon by a standard surface micromachining process. When an input current is applied, self-heating of the beams causes high compressive stresses and the beam buckles. The buckling behavior is found to be continuous and controllable. This continuous, controllable mechanism may be used as microactuators in microsystems. The electro-thermal responses of suspended micro beams had been established previously [7]. This paper continues efforts in the investigation of thermal-elastic buckling responses to complete the electro-thermal-elastic characterization of micromachined beams.

The problem of buckling has been investigated for many years; most previous efforts have been concentrated on large deflection of statically determined beams [8–10], or the Elastica problem coupled with non-linear constitutive equations [11,12]. In this paper, buckling was the result of electrical heating and thermal stresses instead of pure forces. This thermal-elastic problem is investigated in this paper by the coupled problem of Elastica [13,14] and the Duhamel–Neumann law (e.g. [15]).

2. Theoretical analysis

2.1. Electro-thermal modeling

From previous works [7], the average temperature of a lineshape suspended micro beam under an input current is derived as:

$$T_{avg} = T_r - (T_r - T_\infty)\frac{\tanh\left(\sqrt{\varepsilon}\frac{L}{2}\right)}{\sqrt{\varepsilon}\frac{L}{2}} \qquad (1)$$

where T_r and ε are functions of the applied current, geometry, thermal properties and the excessive heat flux shape factor [7].

2.2. Problem of Elastica

As illustrated in Fig. 1, when a clamped–clamped beam is subjected to a temperature rise, two thermal loads, P, are assumed to have the same magnitude but with opposite directions at both ends of the beam. The relation between the thermal load and deflection angles along the axial axis can be derived by the classical problem of Elastica [13,14]. However, unlike statically determined problems (such as cantilevers) which have been studied and solved previously [8, 9, 16], our case is a statically indeterminate one. Special procedures

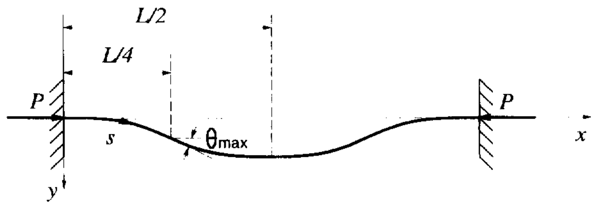

Fig. 1. The schematic diagram of a buckled beam under thermal loads.

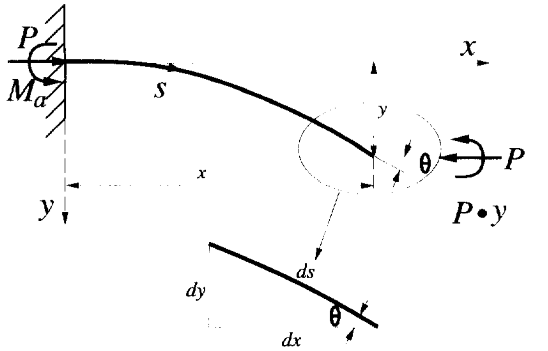

Fig. 2. Free body diagram of a deflected beam.

have to be taken before this problem can be solved. Theoretically, the shape of the beam after buckling is symmetrical such that only one-quarter of the beam is analyzed. The governing equation for the load and deflection relation is derived from the moment–curvature diagram illustrated in Fig. 2, where θ denotes the deflection angle with respect to the original horizontal direction, x, of the beam. The moment–curvature relation is derived as:

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{-Py - M_a}{EI} \tag{2}$$

in which s is the coordinate along the deflected beam, M_a is an unknown reaction moment at the clamped edge, y is the deflection, and E and I are the Young's modulus and moment of inertia of the beam, respectively. The coordinates x, y, s and θ have the following relations:

$$\frac{\mathrm{d}y}{\mathrm{d}s} = \sin\theta \tag{3}$$

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \cos\theta \tag{4}$$

Further differentiating eq. (2) with respect to s results in the governing equation of the Elastica problem:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + \frac{P}{EI}\sin\theta = 0 \tag{5}$$

There are four boundary conditions:

$$\theta|_{s=\frac{L'}{4}} = \theta_{\max} \tag{6}$$

$$\theta|_{s=0} = 0 \tag{7}$$

$$\left.\frac{\mathrm{d}\theta}{\mathrm{d}s}\right|_{s=\frac{L'}{4}} = 0 \tag{8}$$

$$\left.\frac{\mathrm{d}\theta}{\mathrm{d}s}\right|_{s=0} = \frac{-M_a}{EI} \tag{9}$$

where $\theta_{\max}$ is the maximum deflection angle at the point of one-quarter length of the beam. L' is the total length of the deformed beam and its magnitude can be derived by rearranging and integrating eq. (5) twice. The term is then solved by using the boundary conditions of eq. (6) and eq. (8) [17].

$$L' = \frac{4}{\sqrt{\frac{P}{EI}}}K(\beta) \tag{10}$$

$$\beta = \sin\frac{\theta_{\max}}{2} \tag{11}$$

where $K(\beta)$ is the complete elliptic integral of the first kind. The original length, L, is the integration over x and can also be represented as a function of $\theta_{\max}$ [17]:

$$L = \frac{4}{\sqrt{\frac{P}{EI}}}(2E(\beta) - K(\beta)) \tag{12}$$

where $E(\beta)$ is the complete elliptic integral of the second kind.

2.3. Analytic solutions

In order to determine the shape of deflected beams under a thermal load, P, two cases are considered: the infinitesimal deflection and finite deflection. When the deflection is infinitesimal, θ is close to zero.

$$\lim_{\theta\to 0}\int \mathrm{d}s = L = L' \tag{13}$$

$$\lim_{\theta \to 0} K(\beta) = \frac{\pi}{2} \tag{14}$$

By substituting eqs. (13) and (14) into eq. (10), the buckling load P_{cr} is derived as:

$$P_{cr} = \frac{4\pi^2 EI}{L^2} \tag{15}$$

This relation is identical to the result derived from the theorem of Euler buckling of columns [18, 14].

If the deflection is not infinitesimal, the finite deflection can be formulated as:

$$\int \mathrm{d}y = \frac{1}{\sqrt{\frac{2P}{EI}}} \int_0^{\theta} \frac{\sin\theta \, \mathrm{d}\theta}{\pm\sqrt{\cos\theta - \cos\theta_{max}}} \tag{16}$$

Owing to the symmetry and antisymmetry of the deflection shape, the integration can be separated into two parts:

$$\bullet 0 < x < \frac{L}{4} \tag{17}$$

$$y = \frac{1}{\sqrt{\frac{2P}{EI}}} \int_0^{\theta} \frac{\sin\theta \, \mathrm{d}\theta}{\sqrt{\cos\theta - \cos\theta_{max}}}$$

$$= \frac{2\beta}{\sqrt{\frac{P}{EI}}} (1 - \cos\phi)$$

$$\bullet \frac{L}{4} < x < \frac{L}{2} \tag{18}$$

$$y = \frac{1}{\sqrt{\frac{2P}{EI}}} \left(\int_0^{\theta_{max}} \frac{\sin\theta \, \mathrm{d}\theta}{\sqrt{\cos\theta - \cos\theta_{max}}} + \int_{\theta_{max}}^{\theta} \frac{-\sin\theta \, \mathrm{d}\theta}{\sqrt{\cos\theta - \cos\theta_{max}}} \right)$$

$$= \frac{2\beta}{\sqrt{\frac{P}{EI}}} (1 + \cos\phi)$$

where

$$\phi = \sin^{-1}\left(\frac{\sin\frac{\theta}{2}}{\beta}\right) \tag{19}$$

The maximum deflection occurs at the middle of the beam.

$$y_{max} = \frac{4\beta}{\sqrt{\frac{P}{EI}}} \tag{20}$$

It is observed that y_{max} is a function of the thermal load, P, and the maximum deflection angle θ_{max}.

2.4. Thermal-elastic modeling

The Duhamel–Neumann law is used for the thermal-elastic analysis [15].

$$\tau_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} - B_{ij}(T - T_0) \tag{21}$$

where T is the average temperature of the beam, T_0 is a reference temperature, τ_{ij} are the stress components, λ and μ are the Lamè constants, ε_{ij} are the strain components, and B_{ij} are the numerical constants in deriving the Duhamel–Neumann law. Polysilicon is treated as an isotropic material and

$$B_{ij} = \delta_{ij}B = \delta_{ij}\frac{\alpha E}{1 - 2v} \tag{22}$$

where v is the Poisson's ratio and α is the thermal expansion coefficient. Furthermore, since only the axial direction is under thermal loads, the stresses in the other two directions are considered as released. The Duhamel–Neumann law can be simplified to a one-dimensional problem:

$$\tau = E\varepsilon - \alpha E(T - T_0) \tag{23}$$

where E is the Young's modulus and

$$\tau = \frac{P}{A} \tag{24}$$

$$\varepsilon = \frac{L' - L}{L} \tag{25}$$

If the deflected length L' is replaced by θ_{max} and other material properties in eq. (10), the temperature–load relation can be derived as:

$$T = \frac{4K(\beta)EA\sqrt{PEI} - PLEA - P^2L}{PLEA\alpha} + T_0 \tag{26}$$

Substituting eq. (20) into eq. (23), the temperature–maximum deflection relation can be derived:

$$y_{max} = 4\beta\sqrt{\frac{I}{[-\varepsilon + \alpha(T - T_0)]A}} \tag{27}$$

The electro-thermal-elastic problem can then be solved by substituting eq. (1) into eq. (27); the relation between applied current and the consequent maximum deflection can be derived as:

$$y_{max} = 4\beta\sqrt{\frac{I}{\left[-\varepsilon + \alpha\left(T_r - (T_r - T_0)\frac{\tanh\left(\sqrt{\varepsilon}\frac{L}{2}\right)}{\sqrt{\varepsilon}\frac{L}{2}} - T_0\right)\right]A}} \tag{28}$$

This is the analytical electro-thermal-elastic equation for the suspended micro beams.

3. Numerical solutions and discussion

In the above analysis, theoretical electro-thermal-elastic behavior of clamped–clamped beams has been established. Computer simulation is used to obtain the complete solutions. It is observed that under a specified temperature during the thermal post-buckling stage, there is a unique solution of θ_{max} and P for both eq. (12) and eq. (26). Because of the non-linear nature of eq. (26), it is easier to specify θ_{max} and seek the corresponding T, P and other parameters. A micro beam with width of 2 μm, thickness of 2.2 μm and length of 100 μm was used as the simulation sample and the results are listed in Table 1. It is observed that before thermal buckling, θ_{max} is zero. The beam will buckle if the average temperature of the beam is over than 531°C. Since the thickness of the beam is larger than the width, buckling in the lateral direction instead of the vertical direction will occur. The thermal load seems to decrease when the temperature continues to rise as observed in Table 1. This behavior may come from the fact that shape changes of the beam have partially released the thermal stresses. The trend continues as temperature increases until the melting point of the beam is reached.

The shape of the clamped–clamped beam after thermal buckling is one of the most important things to be characterized. This shape can be solved if either θ_{max}, P or T is specified. Figure 3 shows calculation results of two buckled beams at different average temperatures of 900 and 1200°C, respectively. It is noted that the vertical scale is much smaller than the horizontal scale shown in

TABLE 1 Thermal-elastic parameters

θ_{max} (deg)	P/P_{cr}	ε	y_{max} (m)	Temperature (°C)
0.000000	1.000000e+00	0.000000e+00	0.000000e+00	5.311336e+02
1.000000	9.998858e−01	7.615894e−05	5.555802e−07	5.603676e+02
2.000000	9.995432e−01	3.046909e−04	1.111309e−06	6.480911e+02
3.000000	9.989722e−01	6.857613e−04	1.667333e−06	7.943677e+02
4.000000	9.981732e−01	1.219646e−03	2.223803e−06	9.993036e+02
5.000000	9.971464e−01	1.906733e−03	2.780867e−06	1.263048e+03
5.800000	9.961612e−01	2.567038e−03	3.227047e−06	1.516513e+03

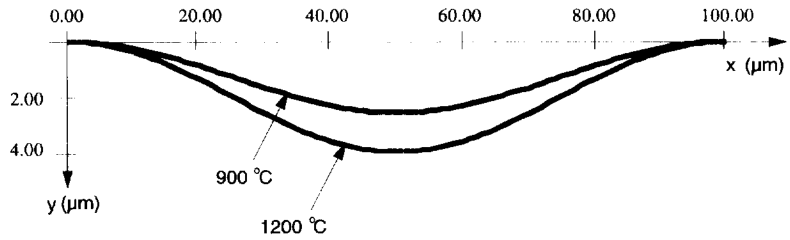

Fig. 3. Deflected shapes of a 100 μm long micro beam under different temperatures.

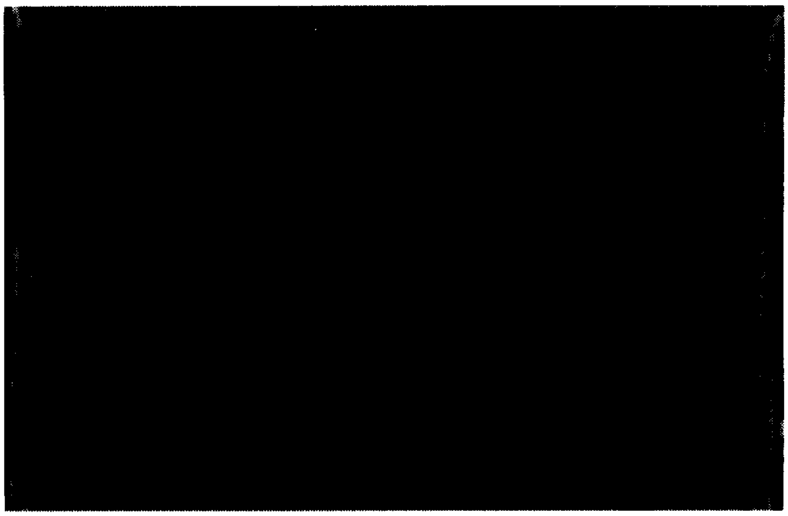

Fig. 4. An optical microphotograph of a suspended beam in the thermal post-buckling stage under a 6 mA input current.

Fig. 3 and the maximum lateral deflection is about 2.7 μm at the center of the beam when the average temperature reaches 900°C. Figure 4 shows the experimental result of a micromachined beam under a 6 mA input current in the thermal post-buckling stage. This microphotograph was taken under an optical microscope which is attached to a probe station. It is observed that the simulation results are consistent with the experimental observations.

In order to calculate the thermal loads with respect to different input currents, pre-buckling and post-buckling states are analyzed separately. Before buckling, the strain is zero and the magnitudes of thermal loads under different temperatures can be derived by using eq. (23). In the post-buckling stage, eq. (26) is used to derive the load–temperature relation.

The electro-thermal-elastic relation can then be found by using eq. (28) as shown in Fig. 5. It is observed that there is no deflection before the input current is high enough to cause buckling. After the beam is buckled, the maximum deflection increases as the input current increases. The highest deflection can be achieved before melting of polysilicon is 3.11 μm under a 7.40 mA input current.

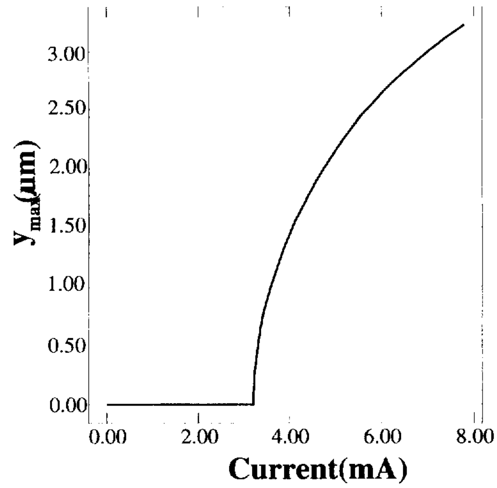

Fig. 5. Simulation results of maximum deflections under different input currents.

4. Conclusion

By solving the coupled problems of Elastica and the Duhamel–Neumann law, a complete solution for clamped–clamped elastic beams under thermal post-buckling is established. The thermal-elastic analysis when combined with the electro-thermal analysis which has been developed previously completes the analytical model for electro-thermal-elastic responses of suspended micro beams. It is found that a micro beam with width $2\,\mu m$, thickness $2.2\,\mu m$ and length $100\,\mu m$ will have a lateral deflection of $2.7\,\mu m$ at the middle of the beam when an input current of $6\,mA$ is applied. Thermally driven miniaturized actuators which utilize electro-thermal post-buckling behavior could take advantage of the electro-thermal-elastic characterizations developed in this paper.

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